

# NOTE ON 2D SCHRÖDINGER OPERATORS WITH $\delta$ -INTERACTIONS ON ANGLES AND CROSSING LINES

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**ABSTRACT.** In this note we sharpen the lower bound from [LLP10] on the spectrum of the 2D Schrödinger operator with a  $\delta$ -interaction supported on a planar angle. Using the same method we obtain the lower bound on the spectrum of the 2D Schrödinger operator with a  $\delta$ -interaction supported on crossing straight lines. The latter operators arise in the three-body quantum problem with  $\delta$ -interactions between particles.

## 1. INTRODUCTION

Self-adjoint Schrödinger operators with  $\delta$ -interactions supported on sufficiently regular hypersurfaces can be defined via closed, densely defined, symmetric and lower-semibounded quadratic forms using the first representation theorem, see [BEKS94] and also [BLL13].

**$\delta$ -interactions on angles.** In our first model the support of the  $\delta$ -interaction is the set  $\Sigma_\varphi \subset \mathbb{R}^2$ , which consists of two rays meeting at the common origin and constituting the angle  $\varphi \in (0, \pi]$  as in Figure 1.

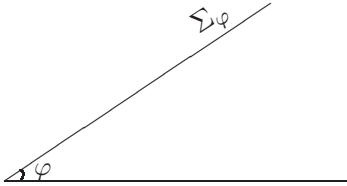


FIGURE 1. The angle  $\Sigma_\varphi$  of degree  $\varphi \in (0, \pi]$ .

The quadratic form in  $L^2(\mathbb{R}^2)$

$$(1.1) \quad \mathfrak{a}_\varphi[f] := \|\nabla f\|_{L^2(\mathbb{R}^2; \mathbb{C}^2)}^2 - \alpha \|f|_{\Sigma_\varphi}\|_{L^2(\Sigma_\varphi)}^2, \quad \text{dom } \mathfrak{a}_\varphi := H^1(\mathbb{R}^2),$$

is closed, densely defined, symmetric and lower-semibounded, where  $f|_{\Sigma_\varphi}$  is the trace of  $f$  on  $\Sigma_\varphi$ , and the constant  $\alpha > 0$  is called the strength of interaction. The corresponding self-adjoint operator in  $L^2(\mathbb{R}^2)$  we denote by  $A_\varphi$ . Known spectral properties of this operator include explicit representation

of the essential spectrum  $\sigma_{\text{ess}}(A_\varphi) = [-\alpha^2/4, +\infty)$  and some information on the discrete spectrum:  $\sharp\sigma_{\text{d}}(A_\varphi) \geq 1$  if and only if  $\varphi \neq \pi$ . These two statements can be deduced from more general results by Exner and Ichinose [EI01]. They are complemented by Exner and Nemčová in [EN03] with the limiting property  $\sharp\sigma_{\text{d}}(A_\varphi) \rightarrow +\infty$  as  $\varphi \rightarrow 0+$ .

In [LLP10] the author obtained jointly with Igor Lobanov and Igor Yu. Popov a general result, which implies the lower bound on the spectrum of  $A_\varphi$

$$(1.2) \quad \inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{4 \sin^2(\varphi/2)}.$$

This bound is close to optimal for  $\varphi$  close to  $\pi$ , whereas in the limit  $\varphi \rightarrow 0+$  the bound tends to  $-\infty$ . In the present note we sharpen this bound. Namely, we obtain

$$(1.3) \quad \inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{(1 + \sin(\varphi/2))^2}.$$

The new bound yields that the operators  $A_\varphi$  are uniformly lower-semibounded with respect to  $\varphi$  and

$$\inf \sigma(A_\varphi) \geq -\alpha^2$$

holds for all  $\varphi \in (0, \pi]$ . This observation agrees well with physical expectations. Note that separation of variables yields that  $\inf \sigma(A_\pi) = -\alpha^2/4$  and in this case the lower bound in (1.3) coincides with the exact spectral bottom.

For sufficiently sharp angles upper bounds on  $\inf \sigma(A_\varphi)$  were obtained by Brown, Eastham and Wood in [BEW08]. See also Open Problem 7.3 in [E08] related to the discrete spectrum of  $A_\varphi$  for  $\varphi$  close to  $\pi$ .

**$\delta$ -interactions on crossing straight lines.** We also consider an analogous model with the  $\delta$ -interaction supported on the set  $\Gamma_\varphi = \Gamma_1 \cup \Gamma_2$ , where  $\Gamma_1$  and  $\Gamma_2$  are two straight lines, which cross at the angle  $\varphi \in (0, \pi)$  as in Figure 2.

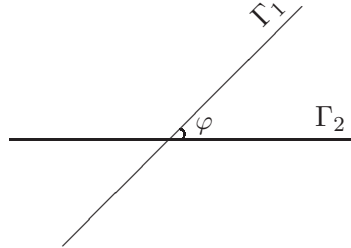


FIGURE 2. The straight lines  $\Gamma_1$  and  $\Gamma_2$  crossing at the angle of degree  $\varphi \in (0, \pi)$ .

The corresponding self-adjoint operator  $B_\varphi$  in  $L^2(\mathbb{R}^2)$  can be defined via the closed, densely defined, symmetric and lower-semibounded quadratic form

$$(1.4) \quad \mathfrak{b}_\varphi[f] := \|\nabla f\|_{L^2(\mathbb{R}^2; \mathbb{C}^2)}^2 - \alpha \|f|_{\Gamma_\varphi}\|_{L^2(\Gamma_\varphi)}^2, \quad \text{dom } \mathfrak{b}_\varphi := H^1(\mathbb{R}^2),$$

in  $L^2(\mathbb{R}^2)$ , where  $\alpha > 0$  is the strength of interaction. According to [EN03] it is known that  $\sigma_{\text{ess}}(B_\varphi) = [-\alpha^2/4, +\infty)$  and that  $\#\sigma_d(B_\varphi) \geq 1$ .

In this note we obtain the lower bound

$$(1.5) \quad \inf \sigma(B_\varphi) \geq -\frac{\alpha^2}{1 + \sin \varphi},$$

using the same method as for the operator  $A_\varphi$ . Separation of variables yields  $\inf \sigma(B_{\pi/2}) = -\alpha^2/2$ , and in this case the lower bound in the estimate (1.5) coincides with the exact spectral bottom.

Upper bounds on  $\inf \sigma(B_\varphi)$  were obtained in [BEW08, BEW09]. The operators of the type  $B_\varphi$  arise in the one-dimensional quantum three-body problem after excluding the center of mass, see Cornean, Duclos and Ricaud [CDR06, CDR08] and the references therein.

We want to stress that our proofs are of elementary nature and we do not use any reduction to integral operators acting on interaction supports  $\Sigma_\varphi$  and  $\Gamma_\varphi$ .

## 2. SOBOLEV SPACES ON WEDGES

In this section  $\Omega \subset \mathbb{R}^2$  is a wedge with the angle of degree  $\varphi \in (0, 2\pi)$ . The Sobolev space  $H^1(\Omega)$  is defined as usual, see [McL, Chapter 3]. For any  $f \in H^1(\Omega)$  the trace  $f|_{\partial\Omega} \in L^2(\partial\Omega)$  is well-defined as in [McL, Chapter 3] and [M87].

**Proposition 2.1.** [LP08, Lemma 2.6] *Let  $\Omega$  be a wedge with angle of degree  $\varphi \in (0, \pi]$ . Then for any  $f \in H^1(\Omega)$  the estimate*

$$\|\nabla f\|_{L^2(\Omega; \mathbb{C}^2)}^2 - \gamma \|f|_{\partial\Omega}\|_{L^2(\partial\Omega)}^2 \geq -\frac{\gamma^2}{\sin^2(\varphi/2)} \|f\|_{L^2(\Omega)}^2$$

*holds for all  $\gamma > 0$ .*

**Proposition 2.2.** [LP08, Lemma 2.8] *Let  $\Omega$  be a wedge with angle of degree  $\varphi \in (\pi, 2\pi)$ . Then for any  $f \in H^1(\Omega)$  the estimate*

$$\|\nabla f\|_{L^2(\Omega; \mathbb{C}^2)}^2 - \gamma \|f|_{\partial\Omega}\|_{L^2(\partial\Omega)}^2 \geq -\gamma^2 \|f\|_{L^2(\Omega)}^2$$

*holds for all  $\gamma > 0$ .*

Propositions 2.1 and 2.2 are variational equivalents of spectral results from [LP08].

### 3. A LOWER BOUND ON THE SPECTRUM OF $A_\varphi$

In the next theorem we sharpen the bound (1.2) using only properties of the Sobolev space  $H^1$  on wedges and some optimization.

**Theorem 3.1.** *Let the self-adjoint operator  $A_\varphi$  be associated with the quadratic form given in (1.1). Then the estimate*

$$\inf \sigma(A_\varphi) \geq -\frac{\alpha^2}{(1 + \sin(\varphi/2))^2}$$

*holds.*

*Proof.* The angle  $\Sigma_\varphi$  separates the Euclidean space  $\mathbb{R}^2$  into two wedges  $\Omega_1$  and  $\Omega_2$  with angles of degrees  $\varphi$  and  $2\pi - \varphi$ , see Figure 3.

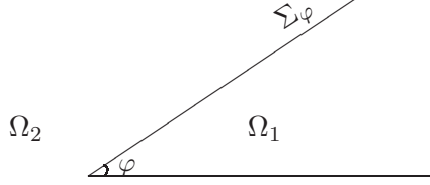


FIGURE 3. The angle  $\Sigma_\varphi$  separates the Euclidean space  $\mathbb{R}^2$  into two wedges  $\Omega_1$  and  $\Omega_2$ .

The underlying Hilbert space can be decomposed as

$$L^2(\mathbb{R}^2) = L^2(\Omega_1) \oplus L^2(\Omega_2).$$

Any  $f \in \text{dom } \mathfrak{a}_\varphi$  can be written as the orthogonal sum  $f_1 \oplus f_2$  with respect to that decomposition of  $L^2(\mathbb{R}^2)$ . Note that  $f_1 \in H^1(\Omega_1)$  and that  $f_2 \in H^1(\Omega_2)$ . Clearly,

$$(3.1) \quad \begin{aligned} \|f\|_{L^2(\mathbb{R}^2)}^2 &= \|f_1\|_{L^2(\Omega_1)}^2 + \|f_2\|_{L^2(\Omega_2)}^2, \\ \|\nabla f\|_{L^2(\mathbb{R}^2; \mathbb{C}^2)}^2 &= \|\nabla f_1\|_{L^2(\Omega_1; \mathbb{C}^2)}^2 + \|\nabla f_2\|_{L^2(\Omega_2; \mathbb{C}^2)}^2. \end{aligned}$$

The coupling constant can be decomposed as  $\alpha = \beta + (\alpha - \beta)$  with some optimization parameter  $\beta \in [0, \alpha]$  and the relation

$$(3.2) \quad \alpha \|f|_{\Sigma_\varphi}\|_{L^2(\Sigma_\varphi)}^2 = \beta \|f_1|_{\partial\Omega_1}\|_{L^2(\partial\Omega_1)}^2 + (\alpha - \beta) \|f_2|_{\partial\Omega_2}\|_{L^2(\partial\Omega_2)}^2$$

holds. According to Proposition 2.1

$$(3.3) \quad \|\nabla f_1\|_{L^2(\Omega_1; \mathbb{C}^2)}^2 - \beta \|f_1|_{\partial\Omega_1}\|_{L^2(\partial\Omega_1)}^2 \geq -\frac{\beta^2}{\sin^2(\varphi/2)} \|f_1\|_{L^2(\Omega_1)}^2,$$

and according to Proposition 2.2

$$(3.4) \quad \|\nabla f_2\|_{L^2(\Omega_2; \mathbb{C}^2)}^2 - (\alpha - \beta) \|f_2|_{\partial\Omega_2}\|_{L^2(\partial\Omega_2)}^2 \geq -(\alpha - \beta)^2 \|f_2\|_{L^2(\Omega_2)}^2.$$

The observations (3.1), (3.2) and the estimates (3.3), (3.4) imply

$$\mathfrak{a}_\varphi[f] \geq -\max \left\{ \frac{\beta^2}{\sin^2(\varphi/2)}, (\alpha - \beta)^2 \right\} \|f\|_{L^2(\mathbb{R}^2)}^2.$$

Making optimization with respect to  $\beta$ , we observe that the maximum between the two values in the estimate above is minimal, when these two values coincide. That is

$$\frac{\beta^2}{\sin^2(\varphi/2)} = (\alpha - \beta)^2,$$

which is equivalent to

$$(3.5) \quad \beta = \frac{\alpha \sin(\varphi/2)}{(1 + \sin(\varphi/2))},$$

resulting in the final estimate

$$\mathfrak{a}_\varphi[f] \geq -\frac{\alpha^2}{(1 + \sin(\varphi/2))^2} \|f\|_{L^2(\mathbb{R}^2)}^2.$$

This final estimate implies the desired spectral bound.  $\square$

*Remark 3.2.* Note that the previously known lower bound (1.2) comes out from the proof of the last theorem if we choose  $\beta = \alpha/2$ , which is the optimal choice in our proof only for  $\varphi = \pi$  as we see from (3.5).

#### 4. A LOWER BOUND ON THE SPECTRUM OF $B_\varphi$

In the next theorem we obtain a lower bound on the spectrum of the self-adjoint operator  $B_\varphi$  using the same idea as in Theorem 3.1.

**Theorem 4.1.** *Let the self-adjoint operator  $B_\varphi$  be associated with the quadratic form given in (1.4). Then the estimate*

$$\inf \sigma(B_\varphi) \geq -\frac{\alpha^2}{1 + \sin \varphi}$$

*holds.*

*Proof.* The crossing straight lines  $\Gamma_1$  and  $\Gamma_2$  separate the Euclidean space  $\mathbb{R}^2$  into four wedges  $\{\Omega_k\}_{k=1}^4$ . Namely, the wedges  $\Omega_1$  and  $\Omega_2$  with angles of degree  $\varphi$  and the wedges  $\Omega_3$  and  $\Omega_4$  with angles of degree  $\pi - \varphi$ , see Figure 4.

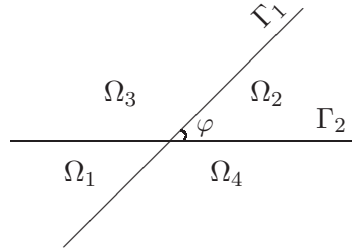


FIGURE 4. The crossing straight lines  $\Gamma_1$  and  $\Gamma_2$  separate the Euclidean space  $\mathbb{R}^2$  into four wedges  $\{\Omega_k\}_{k=1}^4$ .

The underlying Hilbert space can be decomposed as

$$L^2(\mathbb{R}^2) = \bigoplus_{k=1}^4 L^2(\Omega_k).$$

Any  $f \in \text{dom } \mathfrak{b}_\varphi$  can be written as the orthogonal sum  $\bigoplus_{k=1}^4 f_k$  with respect to that decomposition of  $L^2(\mathbb{R}^2)$ . Note that  $f_k \in H^1(\Omega_k)$  for  $k = 1, 2, 3, 4$ . Clearly,

$$(4.1) \quad \|f\|_{L^2(\mathbb{R}^2)}^2 = \sum_{k=1}^4 \|f_k\|_{L^2(\Omega_k)}^2, \quad \|\nabla f\|_{L^2(\mathbb{R}^2; \mathbb{C}^2)}^2 = \sum_{k=1}^4 \|\nabla f_k\|_{L^2(\Omega_k; \mathbb{C}^2)}^2.$$

The coupling constant can be decomposed as  $\alpha = \beta + (\alpha - \beta)$  with some optimization parameter  $\beta \in [0, \alpha]$  and the relation

$$(4.2) \quad \begin{aligned} \alpha \|f|_{\Gamma_\varphi}\|_{L^2(\Gamma_\varphi)}^2 &= \beta \|f_1|_{\partial\Omega_1}\|_{L^2(\partial\Omega_1)}^2 + \beta \|f_2|_{\partial\Omega_2}\|_{L^2(\partial\Omega_2)}^2 \\ &\quad + (\alpha - \beta) \|f_3|_{\partial\Omega_3}\|_{L^2(\partial\Omega_3)}^2 + (\alpha - \beta) \|f_4|_{\partial\Omega_4}\|_{L^2(\partial\Omega_4)}^2 \end{aligned}$$

holds. According to Proposition 2.1

$$(4.3) \quad \begin{aligned} \|\nabla f_1\|_{L^2(\Omega_1; \mathbb{C}^2)}^2 - \beta \|f_1|_{\partial\Omega_1}\|_{L^2(\partial\Omega_1)}^2 &\geq -\frac{\beta^2}{\sin^2(\varphi/2)} \|f_1\|_{L^2(\Omega_1)}^2, \\ \|\nabla f_2\|_{L^2(\Omega_2; \mathbb{C}^2)}^2 - \beta \|f_2|_{\partial\Omega_2}\|_{L^2(\partial\Omega_2)}^2 &\geq -\frac{\beta^2}{\sin^2(\varphi/2)} \|f_2\|_{L^2(\Omega_2)}^2. \end{aligned}$$

Also according to Proposition 2.1

$$(4.4) \quad \begin{aligned} \|\nabla f_3\|_{L^2(\Omega_3; \mathbb{C}^2)}^2 - (\alpha - \beta) \|f_3|_{\partial\Omega_3}\|_{L^2(\partial\Omega_3)}^2 &\geq -\frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)} \|f_3\|_{L^2(\Omega_3)}^2, \\ \|\nabla f_4\|_{L^2(\Omega_4; \mathbb{C}^2)}^2 - (\alpha - \beta) \|f_4|_{\partial\Omega_4}\|_{L^2(\partial\Omega_4)}^2 &\geq -\frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)} \|f_4\|_{L^2(\Omega_4)}^2. \end{aligned}$$

The observations (4.1), (4.2) and the estimates (4.3), (4.4) imply

$$\mathfrak{b}_\varphi[f] \geq -\max \left\{ \frac{\beta^2}{\sin^2(\varphi/2)}, \frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)} \right\} \|f\|_{L^2(\mathbb{R}^2)}^2.$$

Making optimization with respect to  $\beta$ , we observe that the maximum between the two values in the estimate above is minimal, when these two values coincide. That is

$$\frac{\beta^2}{\sin^2(\varphi/2)} = \frac{(\alpha - \beta)^2}{\cos^2(\varphi/2)},$$

which is equivalent to

$$(4.5) \quad \beta = \frac{\alpha \tan(\varphi/2)}{(1 + \tan(\varphi/2))},$$

resulting in the final estimate

$$\mathfrak{b}_\varphi[f] \geq -\frac{\alpha^2}{1 + \sin(\varphi)} \|f\|_{L^2(\mathbb{R}^2)}^2.$$

This final estimate implies the desired spectral bound.  $\square$

*Remark 4.2.* The result of Theorem 4.1 complements [CDR08, Theorem 4.6 (iv)], where the bound

$$\inf \sigma(B_\varphi) \geq -\alpha^2.$$

for all  $\varphi \in (0, \pi)$  was obtained.

## 5. ACKNOWLEDGEMENTS

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